Uniqueness results for the Minkowski problem extended to hedgehogs

by

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Abstract. *The classical Minkowski problem has a natural extension to hedgehogs, that is to Minkowski differences of closed convex hypersurfaces. This extended Minkowski problem is much more difficult since it essentially boils down to the question of solutions of certain Monge-Ampère equations of mixed type on the unit sphere* $\mathbb{S}^n$ *of* $\mathbb{R}^{n+1}$. *In this paper, we mainly consider the uniqueness question and give first results.*

0. Introduction

The classical Minkowski problem is that of the existence, uniqueness and regularity of closed convex hypersurfaces of the Euclidean linear space $\mathbb{R}^{n+1}$ whose Gaussian curvature (in the sense of Gauss' definition) is prescribed as a function of the outer normal vector. In the last century, this fundamental problem played an important role in the development of the theory of elliptic Monge-Ampère equations. Indeed, for C^2_+-hypersurfaces (C^2-hypersurfaces with positive Gaussian curvature), this Minkowski problem is equivalent to the question of solutions of certain Monge-Ampère equations of elliptic type on the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$.

Using approximation by convex polyhedra, Minkowski proved the existence of a weak solution [15]: If K is a continuous positive function on $\mathbb{S}^n$ satisfying the following integral condition

$$\int_{\mathbb{S}^n} \frac{u}{K(u)}\, d\sigma(u) = 0,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^n$, then K is the Gaussian curvature of a unique (up to translation) closed convex hypersurface $\mathcal{H}$. The uniqueness comes from the equality condition in a Minkowski's inequality (e.g. [18, p. 397]). The strong solution is due to Pogorelov [17] and Cheng and Yau [4] who proved independently that: if K is of class C^m, ($m \geq 3$), then the support function of $\mathcal{H}$ is of class $C^{m+1,\alpha}$ for every $\alpha \in]0, 1[$.

Keywords. Minkowski problem, Monge-Ampère equations, convex surfaces, hedgehogs

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This classical Minkowski problem has a natural extension to hedgehogs, that is to Minkowski differences $\mathcal{H}=\mathcal{K}-\mathcal{L}$ of closed convex hypersurfaces $\mathcal{K},\mathcal{L}\in\mathbb{R}^{n+1}$, at least if we restrict ourselves to hypersurfaces whose support functions are C^2. Indeed, the inverse of the Gaussian curvature of such a hedgehog is well-defined and continuous all over $\mathbb{S}^n$ (see Section 1), so that the following existence question arises naturally:

(Q_1) *Existence of a C^2-solution: What are necessary and sufficient conditions for a real continuous function $R\in C(\mathbb{S}^n;\mathbb{R})$ to be the curvature function (that is, the inverse $\frac{1}{K}$ of the Gaussian curvature K) of some hedgehog $\mathcal{H}=\mathcal{K}-\mathcal{L}$?*

Now let us expound the uniqueness question. As we shall see later, for any $h\in C^2(\mathbb{S}^2;\mathbb{R})$, the functions $-h$ and h are the respective support functions of two hedgehogs $\mathcal{H}_{-h}$ and $\mathcal{H}_h$ of $\mathbb{R}^3$ that have the same curvature function and are such that

$$\mathcal{H}_{-h}=s(\mathcal{H}_h),$$

where s is the symmetry with respect to the origin of $\mathbb{R}^3$. Here, we have to recall that noncongruent hedgehogs of $\mathbb{R}^3$ may have the same curvature function [14]: for instance, the two smooth (but not analytic) functions f, g defined on $\mathbb{S}^2$ by

$$f(u):=\begin{cases}\exp(-1/z^2) & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}\quad\text{and}\quad g(u):=\begin{cases} sign(z)\,f(u) & \text{if } z\neq 0\\ 0 & \text{if } z=0,\end{cases}$$

where $u=(x,y,z)\in\mathbb{S}^2$, are the support functions of two noncongruent hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ of $\mathbb{R}^3$ having the same curvature function $R:=1/K\in C(\mathbb{S}^2;\mathbb{R})$, (cf. Figure 1).

Consequently, we state the uniqueness question as follows:

(Q_2) *Uniqueness of a C^2-solution: Let $R\in C(\mathbb{S}^n;R)$ be the curvature function of some hedgehog $\mathcal{H}$. What are necessary and sufficient conditions on R for $\mathcal{H}$ to be uniquely determined (up to parallel translations and central symmetries, the coorienting normal vector being preserved point by point) by R ?*

In particular, it would be very interesting to know whether there exists any pair of noncongruent analytic hedgehogs of $\mathbb{R}^3$ with the same curvature function (by 'analytic hedgehogs' we mean 'hedgehogs with

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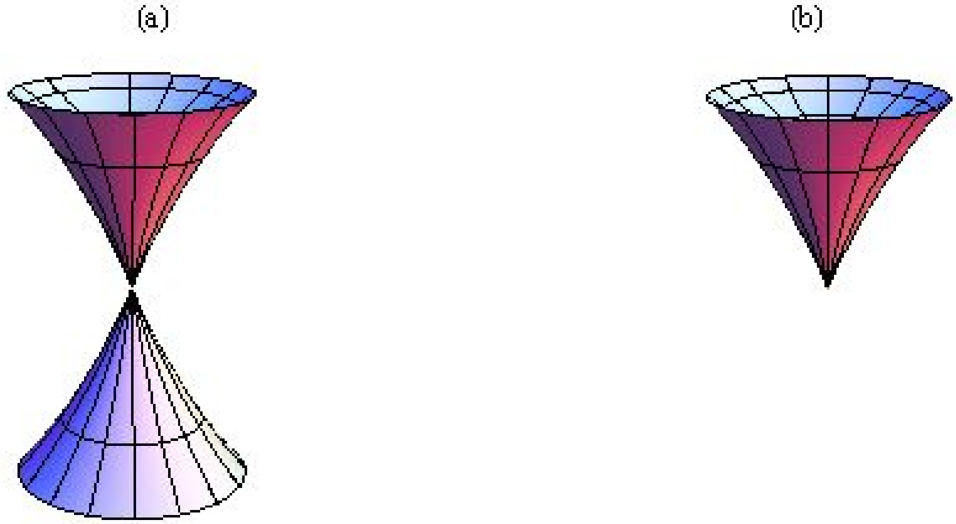

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Figure 1: **Noncongruent hedgehogs with the same curvature function**

an analytic support function'). We shall see in Section 1 that this latest question presents similarities to the open question of knowing whether there exists any pair of noncongruent isometric analytic closed surfaces in $\mathbb{R}^3$.

For $n = 1$, the problem is linear and so can be solved without difficulty [13]. But for $n = 2$, the problem is already very difficult: if $R \in C(\mathbb{S}^2; \mathbb{R})$ changes sign on $\mathbb{S}^2$, the question of existence, uniqueness and regularity of a hedgehog of which R is the curvature function boils down to the study of a Monge-Ampère equation of mixed type, a class of equations for which there is no global result but only local ones by Lin [7] and Zuily [20]. In the present paper, we are mainly interested in the uniqueness question. Question (Q_2) is too difficult to be solved at the present time and our main purpose will be simply to provide conditions under which two hedgehogs of $\mathbb{R}^3$ have distinct curvature functions.

Let H_3 be the $\mathbb{R}$-linear space of C^2-hedgehogs defined up to a translation in $\mathbb{R}^3$ (by 'C^2-hedgehogs' we mean 'hedgehogs with a C^2 support function'). Our first result will be the following.

Theorem. *Let $\mathcal{H}$ and $\mathcal{H}'$ be C^2-hedgehogs that are linearly independent in H_3. If some linear combination of $\mathcal{H}$ and $\mathcal{H}'$ is of class C^2_+, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

As we shall recall in Section 2, any hedgehog can be uniquely split into the sum of its centered and projective parts. Our second result relies on this decomposition of hedgehogs into their centered and projective parts.

Theorem. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two C^2-hedgehogs that are linearly independent in H_3 and the centered parts of which are nontrivial (i.e., distinct from a point) and proportional to one and the same convex surface of class C^2_+. Then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

An immediate consequence will be that:

Corollary. *Two C^2-hedgehogs of nonzero constant width that are linearly independent in H_3 have distinct curvature function.*

Our last result relies on the extension to hedgehogs of the notion of mixed curvature function, which will be recalled in Section 1.

Theorem. *Let $\mathcal{H}$ and $\mathcal{H}'$ be analytic (resp. projective C^2) hedgehogs of $\mathbb{R}^3$ that are linearly independent in H_3. If the mixed curvature function of $\mathcal{H}$ and $\mathcal{H}'$ does not change sign on $\mathbb{S}^2$, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

In Section 1, we shall begin by recalling some basic definitions and facts. Later, we shall present what is already known on the Minkowski problem extended to hedgehogs. Lastly, we shall see different ways of constructing pairs of non-congruent hedgehogs having the same curvature function.

Section 2 will be devoted to the statement of the main results and Section 3 to the proofs and further remarks.

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1. Basic facts and observations on the extended Minkowski problem

As is well-known, every convex body $K \subset \mathbb{R}^{n+1}$ is determined by its support function $h_K : \mathbb{S}^n \longrightarrow \mathbb{R}$, where $h_K(u)$ is defined by $h_K(u) = \sup\{\langle x, u\rangle \,|\, x \in K\}$, $(u \in \mathbb{S}^n)$, that is, as the signed distance from the origin to the support hyperplane with normal vector u. In particular, every closed convex hypersurface of class C^2_+ (i.e., C^2-hypersurface with positive Gaussian curvature) is determined by its support function h (which must be of class C^2 on $\mathbb{S}^n$ [18, p. 111]) as the envelope $\mathcal{H}_h$ of the family of hyperplanes with equation $\langle x, u\rangle = h(u)$. This envelope $\mathcal{H}_h$ is described analytically by the following system of equations

$$\begin{cases} \langle x, u\rangle = h(u) \\ \langle x, .\rangle = dh_u(.) \end{cases}.$$

The second equation is obtained from the first by performing a partial differentiation with respect to u. From the first equation, the orthogonal projection of x onto the line spanned by u is $h(u)\,u$ and from the second one, the orthogonal projection of x onto $u^\perp$ is the gradient of h at u (cf. Figure 2). Therefore, for each $u \in \mathbb{S}^n$, $x_h(u) = h(u)u + (\nabla h)(u)$ is the unique solution of this system.

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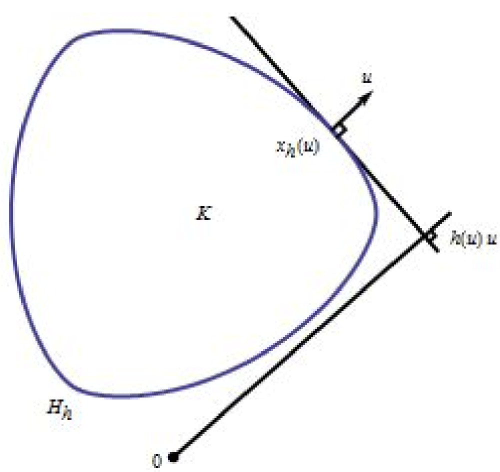

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Figure 2: **Hedgehogs as envelopes parametrized by their Gauss map**

Now, for any C^2-function h on $\mathbb{S}^n$, the envelope $\mathcal{H}_h$ is in fact well-defined (even if h is not the support function of a convex hypersurface). Its natural parametrization $x_h : \mathbb{S}^n \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$ can be interpreted as the inverse of its Gauss map, in the sense that: at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is a normal vector to $\mathcal{H}_h$. We say that $\mathcal{H}_h$ is the hedgehog with support function h (cf. Figure 3). Note that x_h depends linearly on h.

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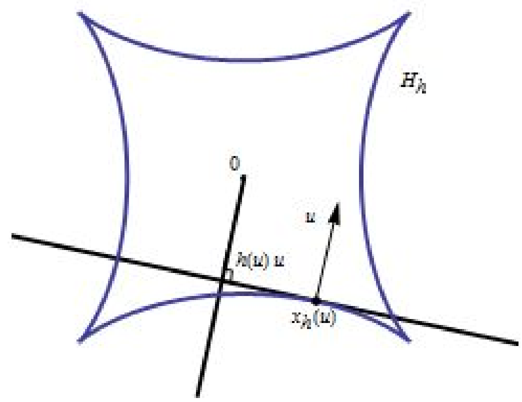

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Figure 3: **A hedgehog with a C^2-support function**

Hedgehogs with a C^2-support function can be regarded as the Minkowski differences of convex hypersurfaces (or convex bodies) of class C_+^2. Indeed [6], given any $h \in C^2(\mathbb{S}^n; \mathbb{R})$, for all large enough real constant r, the functions $h + r$ and r are support functions of convex hypersurfaces of class C_+^2 such that $h = (h + r) - r$.

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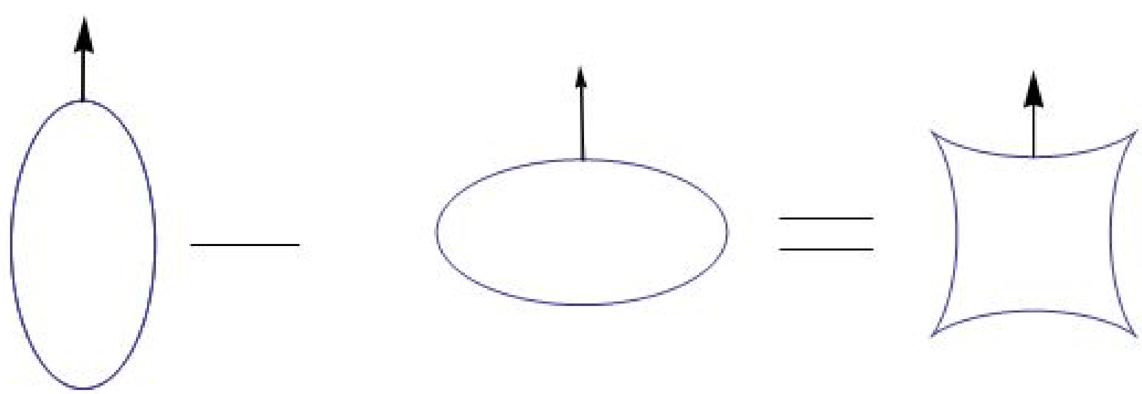

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Figure 4: **Hedgehogs as differences of convex bodies of class C_+^2**

In fact, we can introduce a more general notion of hedgehogs by regarding hedgehogs of $\mathbb{R}^{n+1}$ as Minkowski differences of arbitrary convex bodies of $\mathbb{R}^{n+1}$ [12, 13]. But in the present paper, we shall only consider hedgehogs with a C^2-support function and we will refer to them as 'C^2-hedgehogs'.

Gaussian curvature of C^2-hedgehogs

Let H_{n+1} denote the $\mathbb{R}$-linear space of C^2-hedgehogs defined up to a translation in the Euclidean linear space $\mathbb{R}^{n+1}$ and identified with their support functions. Analytically speaking, saying that a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ is defined up to a translation simply means that the first spherical harmonics of its support function is not specified.

As we saw before, elements of H_{n+1} may be singular hypersurfaces. Since the parametrization x_h can be regarded as the inverse of the Gauss map, the Gaussian curvature K_h of $\mathcal{H}_h$ at $x_h(u)$ is given by $K_h(u) = 1/\det[T_u x_h]$, where $T_u x_h$ is the tangent map of x_h at u. Therefore, singularities are the very points at which the Gaussian curvature is infinite. For every $u \in \mathbb{S}^n$, the tangent map of x_h at the point u is $T_u x_h = h(u)\, Id_{T_u\mathbb{S}^n} + H_h(u)$, where $H_h(u)$ is the symmetric endomorphism associated with the hessian of h at u. Consequently, if λ is an eigenvalue of the hessian of h at u then $\lambda + h(u)$ is (up to the sign) one of the principal radii of curvature of $\mathcal{H}_h$ at $x_h(u)$ and the so-called 'curvature function' $R_h := 1/K_h$ can be given by

$$R_h(u) = \det[H_{ij}(u) + h(u)\,\delta_{ij}], \tag{1}$$

where δ_{ij} are the Kronecker symbols and $(H_{ij}(u))$ the Hessian of h at u with respect to an orthonormal frame on $\mathbb{S}^n$.

The case $n = 2$**.** From (1), the curvature function $R_h := 1/K_h$ of $\mathcal{H}_h \subset \mathbb{R}^3$ is given by $R_h = (\lambda_1 + h)(\lambda_2 + h) = h^2 + h\Delta_2 h + \Delta_{22} h$, where Δ_2 denotes the spherical Laplacian and Δ_{22} the Monge-Ampère operator (respectively the sum and the product of the eigenvalues λ_1, λ_2 of the Hessian of h). So, the equation we shall be dealing with will be the following

$$h^2 + h\Delta_2 h + \Delta_{22} h = 1/K.$$

Note that the so-called 'mixed curvature function' of hedgehogs of $\mathbb{R}^3$, that is,

$$\begin{aligned} R : \mathsf{H}_3^2 &\to C(\mathbb{S}^2; \mathbb{R}) \\ (f, g) &\mapsto R_{(f,g)} := \tfrac{1}{2}(R_{f+g} - R_f - R_g) \end{aligned}$$

is bilinear and symmetric:

$(i)\, \forall (f, g, h) \in \mathsf{H}_3^3, \forall \lambda \in \mathbb{R}, R_{(f+\lambda g, h)} = R_{(f,h)} + \lambda R_{(g,h)}$;

$(ii)\, \forall (f, g) \in \mathsf{H}_3^2, R_{(g,f)} = R_{(f,g)}$.

For any $h \in \mathsf{H}_3$, we have in particular $R_{-h} = R_h$. Note that $R_{(1,f)} = \frac{1}{2}(\Delta_2 h + 2h)$ is (up to the sign) half the sum of the principal radii of curvature of $\mathcal{H}_h \subset \mathbb{R}^3$.

Nonexistence in the Minkowski problem for hedgehogs

The point is that the curvature function $R_h := 1/K_h$ of any C^2-hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is well-defined and continuous all over $\mathbb{S}^n$, including at the singular points of x_h, so that the Minkowski problem arises naturally for hedgehogs. In this paper, we are thus interested in studying the existence and/or uniqueness of C^2-solutions to the Monge-Ampère equation

$$R_h = R, \tag{2}$$

where R is a given real continuous function on $\mathbb{S}^n$.

As in the classical Minkowski problem, the following integral condition is necessary for the existence of such a solution:

$$\int_{\mathbb{S}^n} R(u)\, u d\sigma(u) = 0. \tag{3}$$

It simply expresses that any C^2-hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is a closed hypersurface. But it is no longer sufficient: for instance, the constant function equal to -1 on $\mathbb{S}^2$ satisfies integral condition (3) but it cannot be the curvature function of a hedgehog since there is no compact surface with negative Gaussian curvature in $\mathbb{R}^3$.

This extended Minkowski problem leads to the following examples of Monge-Ampère equations of mixed type for which there is no solution. For every $v \in \mathbb{S}^2$, the smooth function $F_v(u) = 1 - 2\langle u, v\rangle^2$ satisfies integral condition (3) but is not a curvature function on $\mathbb{S}^2$ [11]. In other words, for every fixed $v \in \mathbb{S}^2$, the Monge-Ampère equation $h^2 + h\Delta_2 h + \Delta_{22} h = F_v$ has no C^2-solution on $\mathbb{S}^2$. The proof makes use of orthogonal projection techniques adapted to hedgehogs.

Nonuniqueness in the Minkowski problem for hedgehogs

As recalled in the introduction, two noncongruent hedgehogs of $\mathbb{R}^3$ may have the same curvature function. By bilinearity and symmetry in the arguments of the mixed curvature function $R : \mathsf{H}_3^2 \to C(\mathbb{S}^2; \mathbb{R})$, if $\mathcal{H}_f$ and $\mathcal{H}_g$ are two hedgehogs of $\mathbb{R}^3$ having the same curvature function then, for all $(\lambda, \mu) \in \mathbb{R}^2$, the hedgehogs $\mathcal{H}_{\lambda f + \mu g}$ and $\mathcal{H}_{\mu f + \lambda g}$ also have the same curvature function. For instance, from the pair $\{\mathcal{H}_f, \mathcal{H}_g\}$ of noncongruent hedgehogs represented in Figure 1, we deduce the pair $\{\mathcal{H}_{f+2g}, \mathcal{H}_{2f+g}\}$ of noncongruent hedgehogs (which have the same curvature function) represented in Figure 5.

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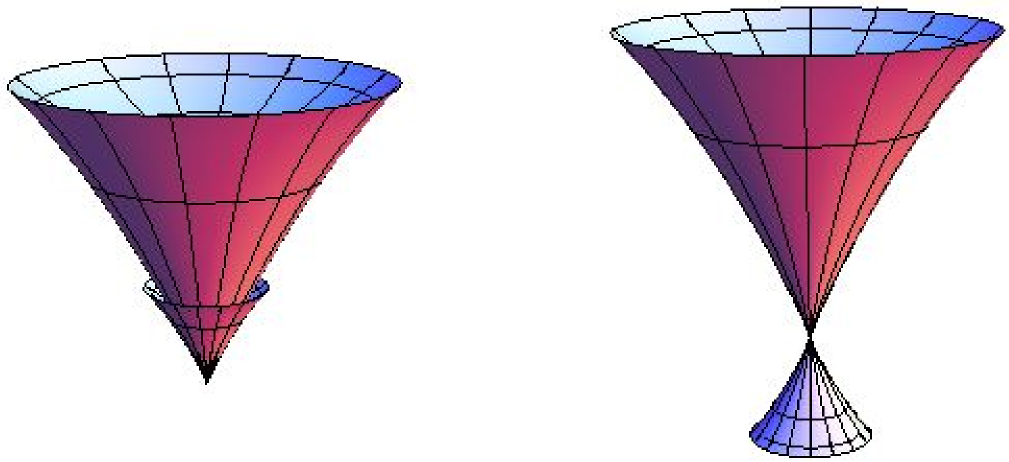

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Figure 5: **Noncongruent hedgehogs with the same curvature function**

A natural but probably difficult question is knowing whether there exists a pair of noncongruent analytic hedgehogs of $\mathbb{R}^3$ with the same curvature function. Let us recall the similar open question of knowing whether there exists a pair of noncongruent isometric analytic closed surfaces in $\mathbb{R}^3$. Smooth closed surfaces can be isometric without being congruent: the usual way of constructing such surfaces is by gluing together smaller congruent pieces. As recalled in [3, p. 131] or [19, p. 366], we can for instance construct a pair $\{S, S'\}$ of noncongruent isometric closed surfaces of revolution as indicated in Figure 6.

We can assume that S admits a parametrization of the form

$$\begin{aligned} x : \mathbb{S}^2 &\rightarrow S \subset \mathbb{R}^3 \\ u &\mapsto \rho(u)\, u, \end{aligned}$$

where ρ is a smooth positive function. Then the hedgehog with support function $h = 1/\rho$ can be regarded as the dual surface of S [8]. This hedgehog $\mathcal{H}_h$ is a surface of revolution whose generating curve (a plane hedgehog which has a fish form) is represented in Figure 7. Replacing the fish's tail by its image under the symmetry with respect to the double point (which by duality corresponds to the plane P) and rotating the plane hedgehog that we get around its axis of symmetry, we generate an other hedgehog which has the same curvature function as $\mathcal{H}_h$ without being congruent to it.

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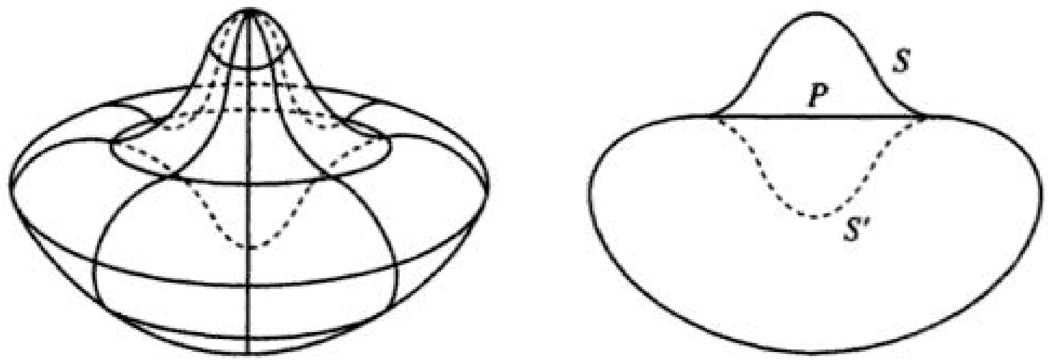

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Figure 6: **Noncongruent isometric surfaces of revolution** [3, p. 131]

2. Statement of results

Recall that H_3 denotes the $\mathbb{R}$-linear space of C^2-hedgehogs defined up to a translation in $\mathbb{R}^3$. Our first result below will be a consequence of the classical Minkowski's uniqueness theorem.

Theorem 1. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be C^2-hedgehogs that are linearly independent in H_3. If some linear combination of $\mathcal{H}_f$ and $\mathcal{H}_g$ is of class C^2_+, then $\mathcal{H}_f$ and $\mathcal{H}_g$ have distinct curvature functions.*

Our second result makes use of the decomposition of hedgehogs into their centered and projective parts.

Decomposition of a hedgehog into its centered and projective parts

Recall that a hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is said to be centered (resp. projective) if its support function h is symmetric (resp. antisymmetric), that is, if we have:

$$\forall u \in \mathbb{S}^n, \quad h(-u) = h(u) \quad (\text{resp. } h(-u) = -h(u)).$$

For instance, the hedgehog $\mathcal{H}_f$ (resp. $\mathcal{H}_g$) of $\mathbb{R}^3$ that is represented in Figure 1.a (resp. Figure 1.b) is centered (resp. projective). Geometri-

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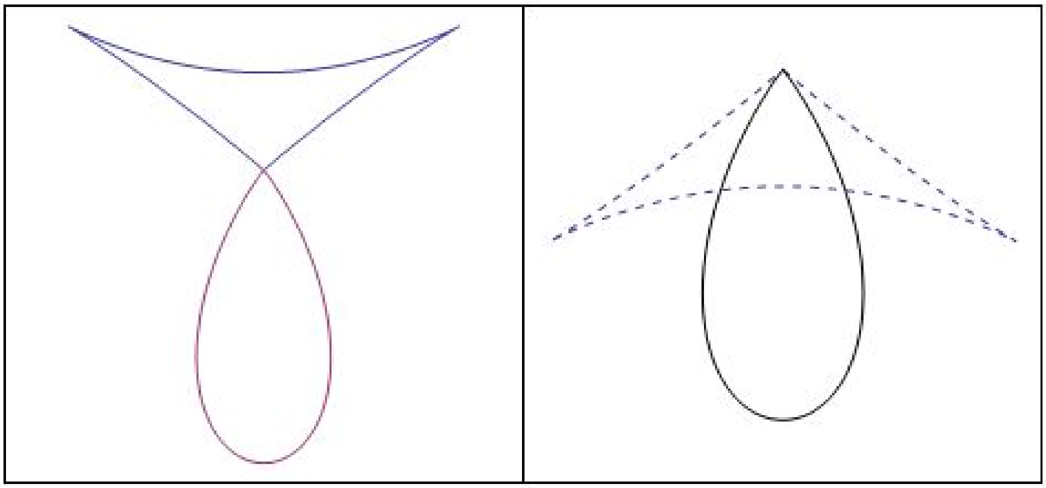

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Figure 7: **Generatrices of revolution hedgehogs with the same curvature**

cally speaking, this means that $\mathcal{H}_h$ is centrally symmetric with respect to the origin (resp. that any pair of antipodal points on the unit sphere $\mathbb{S}^n$ correspond to a same point on the hypersurface $\mathcal{H}_h = x_h(\mathbb{S}^n)$).

Now, the support function h of $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ can be uniquely split into the sum of its symmetric and antisymmetric parts f and g:

$$\forall u \in \mathbb{S}^n, \quad h(u) = f(u) + g(u) \quad \text{where} \quad \begin{cases} f(u) = \frac{1}{2}(h(u) + h(-u)) \\ g(u) = \frac{1}{2}(h(u) - h(-u)) \end{cases}.$$

Consequently, any hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ has a unique representation of the form $\mathcal{H}_f + \mathcal{H}_g$, where $\mathcal{H}_f$ and $\mathcal{H}_g$ are respectively a centered and a projective hedgehog. We say that $\mathcal{H}_f$ and $\mathcal{H}_g$ are respectively the centered and the projective part of $\mathcal{H}_h$.

Theorem 2. *Let $\mathcal{H}_{h_1}$ and $\mathcal{H}_{h_2}$ be C^2-hedgehogs that are linearly independent in H_3 and the centered parts of which are nontrivial (i.e., distinct from a point) and proportional to one and the same convex surface of class C^2_+. Then $\mathcal{H}_{h_1}$ and $\mathcal{H}_{h_2}$ have distinct curvature functions.*

A hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is said to be of constant width if its centered part has a constant support function. In other words, a hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is of constant width if the signed distance between the two cooriented support hyperplanes that are orthogonal to $u \in \mathbb{S}^n$ does not depend on u, that is, if:

$$\exists r \in \mathbb{R}, \forall u \in \mathbb{S}^n, h(u) + h(-u) = 2r.$$

A straightforward consequence of Theorem 3 is the following corollary.

Corollary 3. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be C^2-hedgehogs that are linearly independent in H_3. If $\mathcal{H}_f$ and $\mathcal{H}_g$ are of nonzero constant width, then their curvature functions R_f and R_g are distinct.*

A hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is said to be analytic if its support function h is C^ω on $\mathbb{S}^n$.

Theorem 4. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be analytic (resp. projective C^2) hedgehogs of $\mathbb{R}^3$ that are linearly independent in H_3. If the mixed curvature function of $\mathcal{H}_f$ and $\mathcal{H}_g$ does not change sign on $\mathbb{S}^2$, then $\mathcal{H}_f$ and $\mathcal{H}_g$ have distinct curvature functions.*

3. Proof of the results and further remarks

Proof of Theorem 1. By assumption, there exists $(\lambda, \mu) \in \mathbb{R}^2$ such that the hedgehog $\mathcal{H}_{\lambda f+\mu g}$ is of class C^2_+. We can assume that $|\lambda| \neq |\mu|$.

Let us assume that $R_f = R_g$. We then have:

$$\begin{aligned} R_{\lambda f+\mu g} &= \lambda^2 R_f + \mu^2 R_g + 2\lambda\mu R_{(f,g)} \\ &= \mu^2 R_f + \lambda^2 R_g + 2\mu\lambda R_{(f,g)} \\ &= R_{\mu f+\lambda g}. \end{aligned}$$

As the hedgehog $\mathcal{H}_{\lambda f+\mu g}$ is of class C^2_+, the equality $R_{\lambda f+\mu g} = R_{\mu f+\lambda g}$ implies the existence of an $\varepsilon \in \{-1, 1\}$ such that $\lambda f + \mu g = \varepsilon (\mu f + \lambda g)$ by Minkowski's uniqueness theorem. We thus have $(\lambda - \varepsilon\mu) f = \varepsilon (\lambda - \varepsilon\mu) g$ and hence $f = \varepsilon g$ since $\lambda - \varepsilon\mu \neq 0$. □

Lemma 5. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be two C^2-hedgehogs of $\mathbb{R}^3$. If $u \in \mathbb{S}^2$ is such that $R_g(u) > 0$, then*

$$R_{(f,g)}(u)^2 \geq R_f(u) R_g(u).$$

Proof of the lemma. Define $Q : \mathbb{R} \to \mathbb{R}$ by $Q(t) = R_{f+tg}(u)$. By bilinearity and symmetry of the mixed curvature function, we have:

$$\forall t \in \mathbb{R}, \quad Q(t) = R_f(u) + 2tR_{(f,g)}(u) + t^2 R_g(u).$$

Let us consider the 'reduced' discriminant $\Delta = R_{(f,g)}(u)^2 - R_f(u) R_g(u)$ of the quadratic trinomial $Q(t)$. On the one hand, from the assumption $R_g(u) > 0$, it follows that $Q(t) > 0$ for any large enough t. On the other hand, there exists some $\lambda \in \mathbb{R}$ such that $R_{(1,f+\lambda g)}(u) = R_{(1,f)}(u) + \lambda R_{(1,g)}(u) = 0$ and hence $Q(\lambda) = R_{f+\lambda g}(u) \leq 0$. Therefore $\Delta \geq 0$, which achieves the proof. □

Surprisingly, there exist nontrivial (i.e., distinct from a point) hedgehogs of $\mathbb{R}^3$ whose curvature function is nonpositive all over $\mathbb{S}^2$ [10, 16], which disproves a conjectured characterization of the 2-sphere [1, 5]. However, the support function of such a hedgehog cannot be neither analytic nor antisymmetric on $\mathbb{S}^2$:

Lemma 6 ([*2, 10 Theorem 3*]). *Let $\mathcal{H}_h$ be an analytic (resp. a projective C^2) hedgehog in $\mathbb{R}^3$. If the curvature function R_h of $\mathcal{H}_h$ is nonpositive all over $\mathbb{S}^2$, then $\mathcal{H}_h$ is reduced to a single point.*

Lemma 7. *Let $\mathcal{H}_g$ be a convex hedgehog of class C^2_+ in $\mathbb{R}^3$. Given a projective hedgehog $\mathcal{H}_f$ in $\mathbb{R}^3$, the mixed curvature function $R_{(f,g)}$ is identically zero on $\mathbb{S}^2$ only if $\mathcal{H}_f$ is reduced to a single point, that is, only if f is the restriction to $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$.*

Proof of the lemma. Since $\mathcal{H}_g$ is of class C^2_+, we have

$$R_f(u) R_g(u) \leq R_{(f,g)}(u)^2$$

by Lemma 5. From $R_{(f,g)}(u) = 0$, we then deduce that $R_f \leq 0$ which implies the result by Lemma 6. □

Proof of Theorem 2. By assumption, h_1 and h_2 are of the form

$$\begin{cases} h_1 = f_1 + \lambda_1 k \\ h_2 = f_2 + \lambda_2 k \end{cases},$$

where λ_1, λ_2 are nonzero real numbers, f_1, f_2 the support functions of projective hedgehogs and k the support function of a centered convex surface of class C^2_+. Assume that $R_{h_1} = R_{h_2}$. By bilinearity and symmetry of the mixed curvature function, this gives

$$R_{f_1} + \lambda_1^2 R_k + 2\lambda_1 R_{(f_1,k)} = R_{f_2} + \lambda_2^2 R_k + 2\lambda_2 R_{(f_2,k)}.$$

Splitting into symmetric and antisymmetric parts, we get

$$\begin{cases} R_{f_1} + \lambda_1^2 R_k = R_{f_2} + \lambda_2^2 R_k \\ \\ \lambda_1 R_{(f_1,k)} = \lambda_2 R_{(f_2,k)} \end{cases}.$$

By linearity of the mixed curvature function in the first argument, the second equation is equivalent to $R_{(\lambda_1 f_1 - \lambda_2 f_2, k)} = 0$. By Lemma 7, this implies that $\mathcal{H}_{\lambda_1 f_1 - \lambda_2 f_2}$ is a point and hence that $\mathcal{H}_{\lambda_1 f_1} = \mathcal{H}_{\lambda_2 f_2}$ in H_3. Now, by multiplying each member of the first equation of the previous system by λ_1^2, we get

$$\lambda_1^2 R_{f_1} + \lambda_1^4 R_k = \lambda_1^2 R_{f_2} + \lambda_1^2 \lambda_2^2 R_k,$$

and hence

$$R_{\lambda_1 f_1} - R_{\lambda_1 f_2} = \lambda_1^2 \left(\lambda_1^2 - \lambda_2^2\right) R_k$$

by bilinearity of the mixed curvature function. Therefore,

$$R_{\lambda_2 f_2} - R_{\lambda_1 f_2} = \lambda_1^2 \left(\lambda_1^2 - \lambda_2^2\right) R_k,$$

that is,

$$\left(\lambda_2^2 - \lambda_1^2\right) \left(R_{f_2} - \lambda_1^2 R_k\right) = 0.$$

As $\mathcal{H}_{f_2}$ is projective (resp. $\mathcal{H}_k$ is convex of class C_+^2), we have [9]:

$$\int_{\mathbb{S}^2} R_{f_2} d\sigma \leq 0 \quad \text{and} \quad \int_{\mathbb{S}^2} R_k d\sigma > 0.$$

Therefore, $R_{f_2} \neq \lambda_1^2 R_k$. From the previous equation, we thus get $\lambda_2^2 = \lambda_1^2$, that is:

$$\exists \varepsilon \in \{-1, 1\}, \lambda_2 = \varepsilon \lambda_1.$$

Now, $\lambda_1 \mathcal{H}_{f_1} = \mathcal{H}_{\lambda_1 f_1} = \mathcal{H}_{\lambda_2 f_2} = \lambda_2 \mathcal{H}_{f_2}$ and λ_1, λ_2 are nonzero. Therefore, we have $\mathcal{H}_{f_1} = \varepsilon \mathcal{H}_{f_2}$ in H_3, that is, $\mathcal{H}_{f_2} = \varepsilon \mathcal{H}_{f_1}$ and hence

$$\mathcal{H}_{h_2} = \mathcal{H}_{f_2 + \lambda_2 k} = \mathcal{H}_{f_2} + \lambda_2 \mathcal{H}_k = \varepsilon \left(\mathcal{H}_{f_1} + \lambda_1 \mathcal{H}_k\right) = \varepsilon \mathcal{H}_{h_1} \text{ in } \mathsf{H}_3,$$

which contradicts the fact that $\mathcal{H}_{h_1}$ and $\mathcal{H}_{h_2}$ are linearly independent in H_3. □

Lemma 8. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be two C^2-hedgehogs of $\mathbb{R}^3$. If their curvature functions R_f and R_g are identically equal on $\mathbb{S}^2$, then either $R_{f-g}(u) \leq 0$ or $R_{f+g}(u) \leq 0$ for all $u \in \mathbb{S}^2$.*

Proof of the lemma. Assume that $R_{f-g}(u) > 0$ (resp. $R_{f+g}(u) > 0$). By Lemma 5, we then have

$$R_{(f-g,f+g)}(u)^2 \geq R_{f-g}(u)\, R_{f+g}(u)\,.$$

Now the assumption $R_f = R_g$ implies

$$R_{(f-g,f+g)} = R_f - R_g = 0 \quad \text{and hence} \quad R_{f-g}(u)\, R_{f+g}(u) \leq 0.$$

Therefore $R_{f-g}(u) \leq 0$ (resp. $R_{f+g}(u) \leq 0$). □

Proof of Theorem 4. We prove the result by contraposition. Assume that R_f and R_g are identically equal on $\mathbb{S}^2$. Since $\mathcal{H}_f$ and $\mathcal{H}_g$ are analytic (resp. projective and C^2) hedgehogs of $\mathbb{R}^3$ that are linearly independent in H_3, it follows from Lemma 6 that there must exist $(u,v) \in \mathbb{S}^2 \times \mathbb{S}^2$ such that $R_{f-g}(u) > 0$ and $R_{f+g}(v) > 0$. By Lemma 8, we then deduce that $R_{f+g}(u) \leq 0$ and $R_{f-g}(v) \leq 0$. Now we have $R_{(f,g)} = \frac{1}{4}(R_{f+g} - R_{f-g})$, so that

$$\begin{cases} R_{f-g}(u) > 0 \\ \\ R_{f+g}(u) \leq 0 \end{cases} \quad \text{and} \quad \begin{cases} R_{f+g}(v) > 0 \\ \\ R_{f-g}(v) \leq 0 \end{cases}$$

implies $R_{(f,g)}(u) < 0$ and $R_{(f,g)}(v) > 0$. □

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