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Results in Mathematics

Gauss Rigidity and Volume Preservation Under Preserving Curvature Deformations for Hedgehogs

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Abstract. We consider Gauss rigidity and Gauss infinitesimal rigidity for hedgehogs of $\mathbb{R}^3$ (regarded as Minkowski differences of closed convex surfaces of $\mathbb{R}^3$ with positive Gaussian curvature). Besides, we prove under an appropriate differentiability condition that whenever we perform a deformation of a hedgehog so that its curvature function remains constant, its algebraic volume also remains constant.

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Keywords. Gauss rigidity, Gauss infinitesimal rigidity, hedgehogs, Minkowski problem, algebraic volume.

1. Introduction

In 1813, Cauchy [3] proved (almost rigorously) his famous rigidity theorem: *Any convex polyhedron of* $\mathbb{R}^3$ *is rigid* (that is, no convex polyhedron of $\mathbb{R}^3$ can be continuously deformed so that its faces remain rigid). First examples of flexible polyhedra were discovered by Bricard [2], but these « Bricard's flexible octahedra » are self-intersecting. The question of rigidity of embedded non-convex polyhedra remained open until 1977 when Connelly [5] discovered a first example of flexible sphere-homeomorphic polyhedron. In the late seventies, Connelly and Sullivan formulated the so-called « bellows conjecture » stating that whenever we perform a rigid deformation of a flexible polyhedron P (that is, a continuous deformation of P that changes only its dihedral angles), the volume of P remains constant. The first proof of the bellows conjecture was given by Sabitov [19]. The second proof by Connelly, Sabitov, and Walz [6] followed 2 years later.

Rodriguez and Rosenberg [18] gave a rigidity result for polyhedral hedgehogs of $\mathbb{R}^3$ (Minkowski differences of convex polyhedra of $\mathbb{R}^3$). Three years later, Panina [17] gave examples of flexible « virtual polytopes » (that is polyhedral hedgehogs) of $\mathbb{R}^3$ which are similar to Bricard's flexible octahedra and proved the following refinement of Rodriguez–Rosenberg theorem: *a virtual polytope of* $\mathbb{R}^3$ *with a convex fan is not flexible.*

Dehn [7] proved that any simplicial convex polyhedron P of $\mathbb{R}^3$ is infinitesimally rigid: any non-trivial first order deformation of P induces a variation of its edge lengths. Gauss infinitesimal rigidity of convex polyhedra was stated and proved by Alexandrov [1]: any non-trivial first order deformation of a convex polyhedron P induces a variation of its face areas. See e.g. [8] for details.

Cohn-Vossen [4] proved that smooth closed surfaces of $\mathbb{R}^3$ with everywhere positive Gaussian curvature are rigid. Smooth closed surfaces of $\mathbb{R}^3$ with everywhere positive Gaussian curvature are also infinitesimally rigid [22] (resp. Gauss infinitesimally rigid [21]), that is every isometric infinitesimal deformation of such a surface is trivial (resp. rigid with respect to the Gaussian curvature regarded as a function of the outer unit normal). See e.g. [9, Sections 1 and 2] for details.

In this paper, we consider Gauss rigidity and Gauss infinitesimal rigidity for hedgehogs of $\mathbb{R}^3$ (regarded as Minkowski differences of closed convex surfaces of $\mathbb{R}^3$ with positive Gaussian curvature). As noticed by Izmestiev [8,9], Gauss rigidity (Gauss infinitesimal rigidity) can be interpreted as uniqueness (resp. « infinitesimal » uniqueness) in the Minkowski problem, that is in the problem of prescribing the n^{th} surface area measure of a polytope P of $\mathbb{R}^{n+1}$ on the unit sphere $\mathbb{S}^n$ (resp. the Gaussian curvature of smooth strictly convex closed hypersurface of $\mathbb{R}^{n+1}$ as a function of the outer unit normal). The author already studied the uniqueness part of the Minkowski problem extended to hedgehogs [12,14,15]. In particular, the author presented different ways of constructing pairs of non-congruent hedgehogs that share the same curvature function (i.e., inverse of the Gaussian curvature) [15]. This will allow us to give examples of nontrivial (i.e., distinct from a point) hedgehogs that are not Gauss infinitesimally rigid.

Assume we have a one parameter family of C^2-hedgehogs $(\mathcal{H}_{h_t})_{t\in[0,1]}$, all with the same curvature function (by 'C^2-hedgehogs' we mean 'hedgehogs with a C^2-support function'). We do not know whether they are congruent in $\mathbb{R}^3$. However, we shall prove a theorem of volume preservation under preserving curvature deformations:

Under an appropriate differentiability condition of the family with respect to the parameter, we shall prove that *all the hedgehogs of the family considered have the same algebraic volume.*

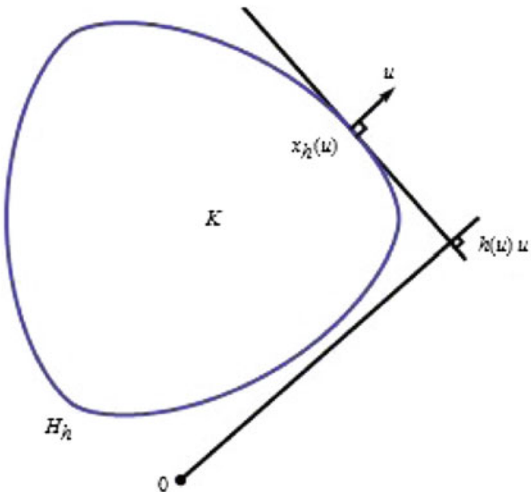

FIGURE 1. Hedgehogs as envelopes parameterized by their Gauss map

2. Basic Definitions C^2- Hedgehogs in $\mathbb{R}^{n+1}$

As is well-known, every convex body $K \subset \mathbb{R}^{n+1}$ is determined by its support function $h_K : \mathbb{S}^n \longrightarrow \mathbb{R}$, where $h_K(u)$ is defined by $h_K(u) = \sup\{\langle x, u\rangle | x \in K\}$, ($u \in \mathbb{S}^n$), that is, as the signed distance from the origin to the support hyperplane with normal vector u. In particular, every closed convex hypersurface of class C^2_+ (i.e., C^2-hypersurface with positive Gaussian curvature) is determined by its support function h (which must be of class C^2 on $\mathbb{S}^n$ [20, p. 111]) as the envelope $\mathcal{H}_h$ of the family of hyperplanes with equation $\langle x, u\rangle = h(u)$. This envelope $\mathcal{H}_h$ is described analytically by the following system of equations

$$\begin{cases} \langle x, u\rangle = h(u) \\ \langle x, .\rangle = dh_u(.) \end{cases} .$$

The second equation is obtained from the first by performing a partial differentiation with respect to u. From the first equation, the orthogonal projection of x onto the line spanned by u is $h(u)u$ and from the second one, the orthogonal projection of x onto $u^\perp$ is the gradient of h at u (cf. Fig. 1). Therefore, for each $u \in \mathbb{S}^n$, $x_h(u) = h(u)u + (\nabla h)(u)$ is the unique solution of this system.

Now, for any C^2-function h on $\mathbb{S}^n$, the envelope $\mathcal{H}_h$ is in fact well-defined (even if h is not the support function of a convex hypersurface). Its natural parametrization $x_h : \mathbb{S}^n \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$ can be interpreted as the inverse of its Gauss map, in the sense that: at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is a normal vector to $\mathcal{H}_h$. We say that $\mathcal{H}_h$ is the hedgehog with support function h (cf. Fig. 2). Note that x_h depends linearly on h.

Hedgehogs with a C^2-support function can be regarded as the Minkow ski differences of convex hypersurfaces of class C^2_+ (Fig. 3). Indeed, given

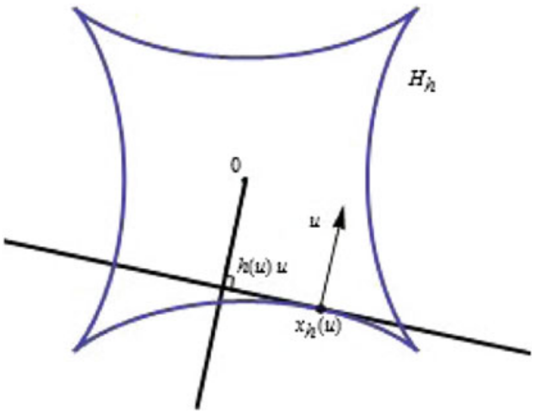

Figure 2. A hedgehog with a C^2-support function

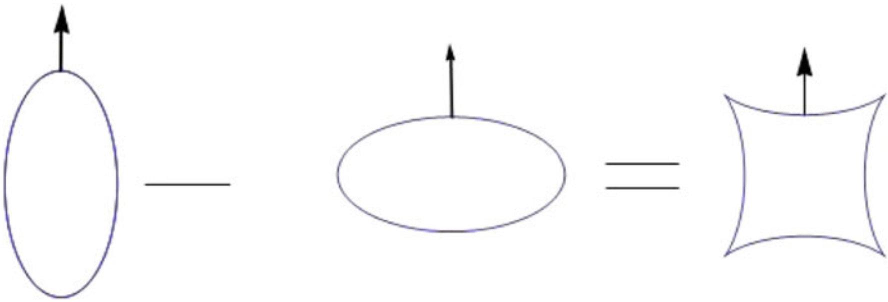

Figure 3. Hedgehogs as differences of convex bodies of class C^2_+

any $h \in C^2(\mathbb{S}^n;\mathbb{R})$, for all large enough real constant r, the functions $h + r$ and r are support functions of convex hypersurfaces of class C^2_+ such that $h = (h + r) - r$.

In fact, we can introduce a more general notion of hedgehogs by regarding hedgehogs of $\mathbb{R}^{n+1}$ as Minkowski differences of arbitrary convex bodies of $\mathbb{R}^{n+1}$ [13]. But in the present paper, we only consider hedgehogs with a C^2-support function and refer to them as 'C^2-hedgehogs'.

3. Gaussian Curvature and Algebraic Volume of C^2 -Hedgehogs

Let H_{n+1} denote the $\mathbb{R}$-linear space of C^2 -hedgehogs defined up to a translation in the Euclidean linear space $\mathbb{R}^{n+1}$ and identified with their support functions. Analytically speaking, saying that a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ is defined up to a translation simply means that the first spherical harmonics of its support function is not specified.

As we saw before, elements of H_{n+1} may be singular hypersurfaces. Since the parametrization x_h can be regarded as the inverse of the Gauss map, the Gaussian curvature K_h of $\mathcal{H}_h$ at $x_h(u)$ is given by $K_h(u) = 1/\det[T_u x_h]$, where $T_u x_h$ is the tangent map of x_h at u. Therefore, singularities are the very points at which the Gaussian curvature is infinite. For every $u \in \mathbb{S}^n$, the tangent map of x_h at the point u is $T_u x_h = h(u) Id_{T_u\mathbb{S}^n} + H_h(u)$, where $H_h(u)$ is the symmetric endomorphism associated with the Hessian of h at$u$. Consequently, if λ is an eigenvalue of the Hessian of h at u then $\lambda + h(u)$ is (up to the sign) one of the principal radii of curvature of $\mathcal{H}_h$ at $x_h(u)$ and the so-called 'curvature function' $R_h := 1/K_h$ can be given by

$$R_h(u) = \det[\nabla_{ij} h(u) + h(u)\delta_{ij}], \tag{1}$$

where δ_{ij} are the Kronecker symbols and $(\nabla_{ij} h(u))$ the Hessian of h at u with respect to an orthonormal frame on $\mathbb{S}^n$.

The case $n = 2$. From (1), the curvature function $R_h := 1/K_h$ of $\mathcal{H}_h \subset \mathbb{R}^3$ is given by $R_h = (\lambda_1 + h)(\lambda_2 + h) = h^2 + h\Delta_2 h + \Delta_{22} h$, where Δ_2 denotes the spherical Laplacian and Δ_{22} the Monge-Ampère operator (respectively the sum and the product of the eigenvalues λ_1, λ_2 of the Hessian of h). Note that the so-called 'mixed curvature function' of hedgehogs of $\mathbb{R}^3$, that is,

$$\begin{aligned} R : \mathsf{H}_3^2 &\to C(\mathbb{S}^2; \mathbb{R}) \\ (f, g) &\mapsto R_{(f,g)} := \frac{1}{2}(R_{f+g} - R_f - R_g) \end{aligned}$$

is bilinear and symmetric:

(*i*) $\forall (f, g, h) \in \mathsf{H}_3^3, \forall \lambda \in \mathbb{R}, R_{(f+\lambda g, h)} = R_{(f,h)} + \lambda R_{(g,h)}$;

(*ii*) $\forall (f, g) \in \mathsf{H}_3^2, R_{(g,f)} = R_{(f,g)}$.

For any $h \in \mathsf{H}_3$, we have in particular $R_{-h} = R_h$. Note that $R_{(1,f)} = \frac{1}{2}(\Delta_2 h + 2h)$ is (up to the sign) half the sum of the principal radii of curvature of $\mathcal{H}_h \subset \mathbb{R}^3$. The (algebraic) volume of a hedgehog $\mathcal{H}_h$ of $\mathbb{R}^3$ is defined by

$$v(h) = \frac{1}{3} \int_{\mathbb{S}^2} h R_h d\sigma,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^2$ and R_h the curvature function [11]. It can be regarded as the integral over $\mathbb{R}^3 - \mathcal{H}_h$ of the index $i_h(x)$ defined as algebraic intersection number of an oriented half-line with origin x with the surface $\mathcal{H}_h$ equipped with its transverse orientation (number independent of the oriented half-line for an open dense set of directions).

4. Gauss Infinitesimal Rigidity in the Context of Hedgehogs

In this work, we shall use the Banach spaces C_m, $(m \in \mathbb{N})$, that were introduced by L. Nirenberg in his study of the Minkowski problem in $\mathbb{R}^3$

[16, p. 380]. The space C_m is defined as follows. The unit sphere $\mathbb{S}^2$ is divided up into three pairs of regions in each of which one of the following coordinate systems is defined:

$$(X, Y) = \left(\frac{x}{z}, \frac{y}{z}\right), (Y, Z) = \left(\frac{y}{x}, \frac{z}{x}\right) \text{ and } (Z, X) = \left(\frac{z}{y}, \frac{x}{y}\right),$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. A function $h : \mathbb{S}^2 \to \mathbb{R}$ belongs to C_m, $(m \in \mathbb{N})$, if in each pair of regions all its partial derivatives (with respect to the corresponding local coordinates) of order less or equal to m exist and are continuous. The norm of every $h \in C_m$ is defined as the sum of the suprema of the absolute values of the partial derivatives up to order m (the suprema being taken with respect to all three pairs of regions).

Definition 1. Let $\mathcal{H}_h$ be a C^2-hedgehog of $\mathbb{R}^3$. A smooth deformation of $\mathcal{H}_h$ is the data of a differentiable map $\widetilde{h} : [0, 1] \to C_2$, $t \mapsto h_t := h(t, .)$ such that $h_0 = h$.

Definition 2. Let $\mathcal{H}_f$ be a C^2-hedgehog of $\mathbb{R}^3$. An infinitesimal isogauss deformation of $\mathcal{H}_f$ is the data of a family $(\mathcal{H}_{f+tg})_{t\in\mathbb{R}}$ of hedgehogs of $\mathbb{R}^3$,

$$\begin{array}{rcl} x_{f+tg} : \mathbb{S}^2 & \to & \mathcal{H}_{f+tg} \subset \mathbb{R}^3 \\ u & \mapsto & x_f(u) + t x_g(u) \end{array}$$

where $\mathcal{H}_g$ is a hedgehog of $\mathbb{R}^3$ such that the mixed curvature function $R_{(f,g)} := \frac{1}{2}(R_{f+g} - R_f - R_g)$ is identically zero on $\mathbb{S}^2$.

Definition 3. Let $\mathcal{H}_f$ be a C^2-hedgehog of $\mathbb{R}^3$. If every infinitesimal isogauss deformation $(\mathcal{H}_{f+tg})_{t\in\mathbb{R}}$ of $\mathcal{H}_f$ is trivial, that is such that $\mathcal{H}_g$ is reduced to a single point, then the hedgehog $\mathcal{H}_f$ will be said to be Gauss infinitesimally rigid.

Remark 1. A hedgehog $\mathcal{H}_g$ is reduced to a single point if, and only if, its support function g is the restriction to $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$, which amounts to saying that its curvature function R_g is identically zero on $\mathbb{S}^2$ [10, Theorem 1]. Therefore, a hedgehog $\mathcal{H}_f$ is Gauss infinitesimally rigid if, and only if, we have:

$$\forall g \in C^2(\mathbb{S}^2; \mathbb{R}), \quad (R_{(f,g)} = 0) \Longrightarrow (R_g = 0).$$

Remark 2. If a hedgehog $\mathcal{H}_f \subset \mathbb{R}^3$ is trivial (that is, reduced to a point), then $\mathcal{H}_f$ is not Gauss infinitesimally rigid. Indeed, for every regular C^2-hedgehog $\mathcal{H}_g \subset \mathbb{R}^3$, we have $R_{(f,g)} = 0$ although R_g is not identically zero on $\mathbb{S}^2$.

5. Gauss Infinitesimal Rigidity of Regular C^2-Hedgehogs of $\mathbb{R}^3$

Let us recall the **proof of the Gauss infinitesimal rigidity** (with respect to the curvature function) **of regular** C^2**- hedgehogs of** $\mathbb{R}^3$ (that are closed convex surfaces of class C_+^2 in $\mathbb{R}^3$). It is essentially a rewriting of the proof by Stoker [21]: Let $\mathcal{H}_f$ be a regular C^2-hedgehog of $\mathbb{R}^3$. Clearly, the regularity of $\mathcal{H}_f$ is equivalent to the strict positivity of its curvature function $R_f := 1/K_f$. If $(\mathcal{H}_{f+tg})_{t\in\mathbb{R}}$ defines an isogauss deformation of $\mathcal{H}_f$, then we have [15, Lemma 5]:

$$0 = R_{(f,g)}^2 \geq R_f . R_g$$

and hence $R_g \leq 0$ on $\mathbb{S}^2$. By taking the origin to be an interior point of the convex body bounded by $\mathcal{H}_f$ in $\mathbb{R}^3$, we may assume without loss of generality that $f > 0$ so that $fRg \leq 0$ on $\mathbb{S}^2$. Now, by symmetry of the mixed volume of hedgehogs of $\mathbb{R}^3$ [11], we get:

$$0 = \int_{\mathbb{S}^2} gR_{(f,g)} d\sigma = \int_{\mathbb{S}^2} fR_{(g,g)} d\sigma = \int_{\mathbb{S}^2} fR_g d\sigma,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^2$. Therefore, R_g is identically zero on $\mathbb{S}^2$ which implies that $\mathcal{H}_g$ is reduced to a single point by Remark 1. $\square$

6. Relation to Minkowski Problem

In the context of hedgehogs, there is a close connection between Gauss infinitesimal rigidity and the uniqueness question in the Minkowski problem. This is due to the following equivalence:

$$\forall (f,g) \in C^2(\mathbb{S}^2;\mathbb{R})^2, \quad (R_f = R_g) \Longleftrightarrow (R_{(f+g,f-g)} = 0).$$

In [14,15], the author gave examples of pairs of non-congruent hedgehogs of $\mathbb{R}^3$ having the same curvature function. From each of these examples, we can deduce examples of nontrivial hedgehogs that are not Gauss infinitesimally rigid. It is for instance the case of the pair of hedgehogs of $\mathbb{R}^3$ given by:

$$f(u) \coloneqq \begin{cases} 0 & \text{if } z \leq 0 \\ \exp(-1/z^2) & \text{if } z > 0 \end{cases} \quad \text{and} \quad g(u) := \begin{cases} \exp(-1/z^2) & \text{if } z < 0 \\ 0 & \text{if } z \geq 0, \end{cases}$$

where $u = (x, y, z) \in \mathbb{S}^2 \subset \mathbb{R}^3$. Indeed, we have clearly $R_{(f,g)} = 0$. Therefore, these two nontrivial hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ are not Gauss infinitesimally rigid. Only nonanalytic examples are known. The question of knowing whether there exists a pair of noncongruent analytic hedgehogs of $\mathbb{R}^3$ with the same curvature function remains open (by 'analytic hedgehogs', we mean 'hedgehogs with an analytic support function'). As a consequence, the question of knowing whether there exist examples of nontrivial analytic hedgehogs that are not Gauss infinitesimally rigid is also open.

7. Volume Preservation Under Curvature Preserving Deformations

Lemma 4. *The curvature function $R : C_2 \to C_0$, $h \mapsto R_h$ is differentiable on C_2, and:*

$$\forall\,(f,g) \in C_2 \times C_2, \qquad dR_f(g) = \lim_{\substack{t\to 0\\ t\neq 0}} \frac{R_{f+tg} - R_f}{t} = 2R_{(f,g)}.$$

Proof of the lemma. Indeed, we have:

$$\begin{aligned}\forall t \in \mathbb{R}_+^*, \quad R_{f+tg} - R_f &= R_f + 2tR_{(f,g)} + t^2 R_g - R_f \\ &= t(2R_{(f,g)} + tR_g),\end{aligned}$$

and hence

$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{R_{f+tg} - R_f}{t} = \lim_{\substack{t\to 0\\ t\neq 0}} (2R_{(f,g)} + tR_g) = 2R_{(f,g)}.$$

Now, we have:

$$\|R_{f+g} - R_f - 2R_{(f,g)}\|_{C_0} = \|R_g\|_{C_0} = o(\|g\|_{C_2}),$$

which achieves the proof. □

Theorem 5. *Let $\mathcal{H}_h$ be a C^2-hedgehog of $\mathbb{R}^3$. If a smooth deformation of $\mathcal{H}_h$, say*

$$\widetilde{h} : [0,1] \to C_2, t \mapsto h_t := h(t,.),$$

preserves the curvature function (that is, is such that $R_{h_t} = R_h$ for all $t \in [0,1]$), then it also preserves the algebraic volume:

$$\forall t \in [0,1], \quad v(h_t) = v(h).$$

Proof of Theorem 5. By assumption, the map $R \circ \widetilde{h} : [0,1] \to C_0$ is constant. Since $\widetilde{h}$ is differentiable by assumption and R by Lemma 4, $R \circ \widetilde{h}$ is differentiable and the chain rule gives:

$$\forall t \in [0,1] \quad (R \circ \widetilde{h})'(t) = 2R_{\left(\widetilde{h}(t), \left(\frac{\partial \widetilde{h}}{\partial t}\right)(t) t\right)}.$$

Therefore, the differentiation yields:

$$\forall t \in [0,1], \quad R_{\left(\widetilde{h}(t), \left(\frac{\partial \widetilde{h}}{\partial t}\right)(t)\right)} = 0. \tag{2}$$

Now, for every $t_0 \in [0,1]$, we have :

$$\forall t \in [0,1] - \{t_0\}, \quad \frac{v(\widetilde{h}(t)) - v(\widetilde{h}(t_0))}{t - t_0} = \frac{1}{3} \int_{\mathbb{S}^2} \frac{\widetilde{h}(t) - \widetilde{h}(t_0)}{t - t_0} R_{\widetilde{h}(t_0)} d\sigma$$

and hence:

$$\lim_{\substack{t\to t_0\\ t\neq t_0}} \frac{v(\widetilde{h}(t))-v(\widetilde{h}(t_0))}{t-t_0} = \frac{1}{3}\int_{\mathbb{S}^2}\left(\frac{\partial \widetilde{h}}{\partial t}\right)(t_0)R_{\widetilde{h}(t_0)}d\sigma.$$

Besides, by symmetry of the mixed volume of hedgehogs [11], we have:

$$\frac{1}{3}\int_{\mathbb{S}^2}\left(\frac{\partial \widetilde{h}}{\partial t}\right)(t_0)R_{\widetilde{h}(t_0)}d\sigma = \frac{1}{3}\int_{\mathbb{S}^2}\widetilde{h}(t_0)R_{\left(\widetilde{h}(t_0),\left(\frac{\partial \widetilde{h}}{\partial t}\right)(t_0)\right)}d\sigma.$$

From (2), we then deduce that:

$$\forall t_0\in[0,1],\quad (v\circ\widetilde{h})'(t_0)=\lim_{\substack{t\to t_0\\ t\neq t_0}} \frac{v(\widetilde{h}(t))-v(\widetilde{h}(t_0))}{t-t_0}=0,$$

and thus all the hedgehogs of the family $(\mathcal{H}_{h_t})$ have the same (algebraic) volume. □

Remark 3. Noncongruent hedgehogs that share the same curvature function may of course have different (algebraic) volumes. It is for instance the case of the hedgehogs shown on Figure 4 whose support functions f, g are defined on $\mathbb{S}^2$ by

$$f(u) := \begin{cases}\exp(-1/z^2) & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases} \quad \text{and} \quad g(u) := \begin{cases} sign(z)f(u) & \text{if } z\neq 0\\ 0 & \text{if } z=0,\end{cases}$$

where $u=(x,y,z)\in\mathbb{S}^2\subset\mathbb{R}^3$.

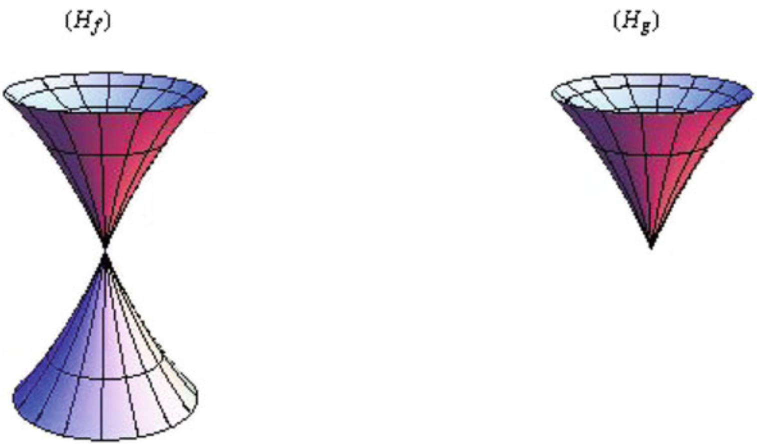

FIGURE 4. Same curvature function and different algebraic volumes

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